Games and Strategies

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1 Introduction

The popularity of games in math olympiads is steadily increasing. They are very appealing, and they can incorporate parts of algebra, combinatorics, geometry, and number theory. A few specific ideas to keep in mind are:

1. Always try small examples. See who wins, try to formulate a conjecture in general, and try to prove the small cases by a non-exhaustion method that may generalize.

2. When there is a reasonable description of the winner based on N, induction is a very good idea to try.

3. Strategy stealing: if you think the first player can win, you can sometimes show that an assumed winning strategy for the second player could be used by the first player instead. Situations where the second player can steal the first player's strategy will be fairly rare. This method is good in problems where an explicit strategy is difficult to describe.

4. If player 1 should win, try to prove that player 2 should win instead, and see what goes wrong (and vice versa). Finding the barrier to winning and having the other player exploit this can lead you to a solution.

2 Examples

Example 1 (TOT 2005)**.** *David and Enrique wish to divide* 25 *coins, of denominations* $1, 2, 3, \ldots, 25$ *pesos. In each move, one of them chooses a coin, and the other player decides who must take this coin. David makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more pesos at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more pesos wins. Which player has a winning strategy?*

Solution. First, the sum of all the coins is $1 + 2 + \cdots + 25 = 325$, whence the players have distinct sums, and one of them wins. We claim that Enrique wins: on the first turn, Enrique chooses which of him or David receives the first coin. These situations are mirror opposites, hence Enrique can win in exactly one of them, and thus he has the winning strategy (this does not give us an explicit strategy to follow however). $\square$

Example 2 (2014 IMOSL C8 special case)**.** *A deck consists of* 1024 *cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared. Show that the first player has a winning strategy.*

Solution. For sets S_1, S_2, denote $S_1 \Delta S_2$ to be the sum of S_1 and S_2, which is the set of elements in either S_1 or S_2 but not both. In particular, $S_1 \Delta S_2 \Delta \cdots \Delta S_n$ is the set of elements in an odd number of the S_i's. As each digit appears on exactly 512 cards, the sum of all cards is the empty set. Let the sum of the cards of the first player be W, and then this must be the sum of the cards of the second player as well! Thus the person holding the card W is the winner (in particular, there is a winner).

Denote the first player by F and the second by S, and assume that the S has a winning strategy. Consider two games G_1, G_2 being played at the same time. In G_1, F takes any card A_1, and let S respond with B_1. In the second game, F takes B_1, and let S take A_2. Then F takes A_2 in the first game, S responds with B_2, and they keep going in this manner. At some point, S will take card $A_i = A_1$ from the second pile, and we pause the game momentarily. In game 1, F holds the cards $A_1, A_2, \ldots, A_i$ and S holds the cards $B_1, B_2, \ldots, B_i$. But in game 2, S holds the cards $A_1, A_2, \ldots, A_i$ and F holds the cards $B_1, B_2, \ldots, B_i$! Thus they hold the same sets of cards in each game, but with players reversed.

Repeat this strategy until the deck eventually runs out. Since S has a winning strategy, they must win both games. However, F's cards in game one are the same as S's cards in game two, which is a contradiction. Therefore F has a winning strategy. $\square$

3 Problems

A-level problems should be early-mid CMO level. B and C problems would be olympiad style and IMO level, with C being generally harder than B. This ordering is somewhat subjective, so don't be surprised if you find some problems to be out of place.

A1 Glen and Scott are playing a two person game with the following rules:

- Initially there is a pile of N stones, with $N \geq 2$.
- The players alternate turns, with Glen going first. On his first turn, Glen must remove at least 1 and at most $N - 1$ stones from the pile.
- If a player removes k stones on their turn, then the other player must remove at least 1 and at most $2k - 1$ stones on their next turn.
- The player who removes the last stone wins the game.

Find all values of N for which Scott has a winning strategy, and explain the strategy.

A2 There are some marbles in a bowl. James, Adán and Daniel each take turns removing one or two marbles from the bowl, with James going first, then Adán, then Daniel, then James again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners.

(a) If the game starts with 5 marbles in the bowl, can Adán and Daniel work together and force James to lose?

(b) The game is played again, this time starting with N marbles in the bowl. For what values of N can Adán and Daniel work together and force James to lose?

A3 Let $n \geq 2$ be an integer. Sarah and Matt play a game concerning a country made of n islands. Exactly two of those n islands have a factory. Initially there is no bridge in the country. Sarah and Matt take turns in the following way. In each turn, the player must build a bridge between two different islands I_1 and I_2 such that:

- I_1 and I_2 are not already connected by a bridge.
- at least one of the two islands I_1 and I_2 is connected by a series of bridges to an island with a factory (or has a factory itself). (Indeed, access to a factory is needed for the construction.)

As soon as a player builds a bridge that makes it possible to go from one factory to the other, this player loses the game. (Indeed, it triggers an industrial battle between both factories.) If Sarah starts, then determine (for each $n \geq 2$) who has a winning strategy. (Note: It is allowed to construct a bridge passing above another bridge.)

A4 Let n be a positive integer. A row of $n + 1$ squares is written from left to right, numbered $0, 1, 2, \ldots, n$. Two frogs, named Stewart and Kathlynn, begin a race starting at square 0. For each second that passes, Stewart and Kathlynn make a jump to the right according to the following rules: if there are at least eight squares to the right of Stewart, then Stewart jumps eight squares to the right. Otherwise, Stewart jumps one square to the right. If there are at least seven squares to the right of Kathlynn, then Kathlynn jumps seven squares to the right. Otherwise, Kathlynn jumps one square to the right. Let $A(n)$ and $B(n)$ respectively denote the number of seconds for Stewart and Kathlynn to reach square n. For example, $A(40) = 5$ and $B(40) = 10$.

(a) Determine an integer $n > 200$ for which $B(n) < A(n)$.

(b) Determine the largest integer n for which $B(n) \leq A(n)$.

A5 We have a pile of 2016 cards and a hat. We take out one card, put it in the hat and then divide the remaining cards into two arbitrary non empty piles. In the next step, we choose one of the two piles, we move one card from this pile to the hat and then divide this pile into two arbitrary non empty piles. This procedure is repeated several times : in the k-th step ($k > 1$) we move one card from one of the piles existing after the step $(k - 1)$ to the hat and then divide this pile into two non empty piles. Is it possible that after some number of steps we get all piles containing three cards each?

B1 One hundred pirates played cards. When the game was over, each pirate calculated the amount he won or lost. The pirates have a gold sand as a currency; each has enough to pay his debt. Gold could only change hands in the following way. Either one pirate pays an equal amount to every other pirate, or one pirate receives the same amount from every other pirate. Prove that after several such steps, it is possible for each winner to receive exactly what he has won and for each loser to pay exactly what he has lost.

B2 There are N piles each consisting of a single nut. Tomas and Thomas play the following game in turn, with Tomas going first. At each move, T(h)omas combines two piles that contain coprime numbers of nuts into a new pile. The T(h)omas who can not make a move, loses. For every $N > 2$ determine which player wins.

B3 Howard shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Eric, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Howard moves the card to the empty spot, without telling Eric; otherwise nothing happens. Then Eric names another card and so on, as many times as he likes, until he says "stop"

(a) Can Eric guarantee that after he says "stop," no card is in its initial spot?

(b) Can Eric guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

B4 Rogelio and Marco play a game on a $n \times n$ chessboard. At the beginning, all squares are white apart from one black corner square containing a rook. Players take turns to move the rook to a white square and recolour the square black. The player who can not move loses. Rogelio goes first. Who has a winning strategy?

B5 Ana Paula and Bruno decided to visit Archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Ana Paula and Bruno are playing during the trip: Ana Paula chooses the first island on which they arrive by plane. Then Bruno chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoevers turn to choose next would be the loser. Prove that Ana Paula could always win, regardless of the way Bruno played and regardless of the way the islands were connected.

C1 Seated in a circle are 11 wizards. A different positive integer not exceeding 1000 is pasted onto the forehead of each. A wizard can see the numbers of the other 10, but not his own. Simultaneously, each wizard puts up either his left hand or his right hand. Then each declares the number on his forehead at the same time. Is there a strategy on which the wizards can agree beforehand, which allows each of them to make the correct declaration?

C2 An integer $n > 1$ is given. Pablo and William mark points on a circle in turn. Pablo goes first and uses purple, while William goes second and uses white. The game is over when each player marks n points. Then each player finds the arc of maximal length with ends of his color, which does not contain any other marked points. A player wins if his arc is longer (if the lengths are equal, or both players have no such arcs, the game ends in a draw). Which player has a winning strategy?

C3 David writes some positive integer on a blackboard. Diego can place pluses between some of its digits; then the Juan Carlos calculates the resulting sum (for example, starting from 123456789 one may obtain $12345+6+789 = 13140$). Diego is allowed to apply this procedure to the resulting number up to ten times. Prove that he can always end up with one-digit number.

C4 There are 100 boxes, each containing either a red cube or a blue cube. Sebastian has a sum of money initially, and places bets on the colour of the cube in each box in turn. The bet can be anywhere from 0 up to everything he has at the time. After the bet has been placed, the box is opened. If Sebastian loses, his bet will be taken away. If he wins, he will get his bet back, plus a sum equal to the bet. Then he moves onto the next box, until he has bet on the last one, or until he runs out of money. What is the maximum factor by which he can guarantee to increase his amount of money, if he knows that the exact number of blue cubes is

(a) 1;

(b) some integer k, $1 < k \leq 100$.

C5 A test consists of 30 true or false questions. After the test, Victor gets his score: the number of answers that aren't rong. A victory is defined to be a perfect score. Victor is allowed to take the test several times, and does not know any answer initially. Can Victor work out a strategy that allows him to be victorious after his

(a) 30^{th} attempt?

(b) 25^{th} attempt?